CONVECTIVE INSTABILITY OF A BINARY MIXTURE, PARTICULARLY IN THE NEIGHBORHOOD OF THE CRITICAL POINT

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Criteria of stationary and oscillatory instability of a plane horizontal layer of a binary mixture bounded by solid surfaces are derived. Results of calculations by the Bubnov-Galerkin method are presented in the form of a stability diagram. It is shown that the criterion of convection onset is altered when the pressure gradient is taken into account in a binary system in equilibrium, particularly for a zero diffusion stream.

A new stability criterion, similar to that of Schwarzschild for a pure fluid, is obtained by taking into account pressure gradient effects on stability in the neighborhood of the line of critical points of mixing in a binary mixture.

1. The conditions of convection onset in a mixture with a nonuniform distribution of temperature and concentration differ considerably from those for a pure fluid. As shown in [1] on the example of stability of a plane vertical layer, two kinds of instability related to monotonic and oscillatory perturbations are possible in a mixture. This conclusion is confirmed in [2, 3]. In all these investigations an exact solution of the nonstationary equations of small perturbations was made possible by the selection of either a simple form for the investigated region or of suitable boundary conditions.

Let us consider the stability of mechanical equilibrium of a plane horizontal layer of a mixture bounded by solid surfaces. The temperature and concentration gradients are assumed to be constant and vertical

$$\nabla T_0 = -A_0\gamma, \qquad \nabla C_0 = -B_0\gamma \tag{1.1}$$

Here γ is the unit vector directed vertically upward along the z-axis.

Eliminating pressure and horizontal velocity components from the dimensionless equation of convection in the mixture, taking into account thermal diffusion and diffusive thermal conductivity [4], and assuming (by virtue of the problem unboundedness in horizontal directions) that the dependence on horizontal coordinates is of the form $e^{i\mathbf{kr}}$, where $\mathbf{k}$ is the two-dimensional wave vector in the xy-plane, we obtain for the amplitudes of vertical velocity $V_z \equiv f(z, t)$, temperature $T' \equiv \tau(z, t)$, and concentration $C' \equiv \equiv \xi(z, t)$

$$(D^2 - k^2)\frac{\partial f}{\partial t} = (D^2 - k^2)^2 f - R_T k^2\tau - R_c k^2\xi$$

$$P_T \frac{\partial \tau}{\partial t} = f + (1 + a_1)(D^2 - k^2)\tau + a_2(D^2 - k^2)\xi \tag{1.2}$$

$$P_c \frac{\partial \xi}{\partial t} = f + \frac{a_1}{a_2}(D^2 - k^2)\tau + (D^2 - k^2)\xi \quad \left(D \equiv \frac{\partial}{\partial z}\right)$$

We select the following units: the height of the (fluid) layer l for length; l^2/ν for time; $\varkappa / l$ for velocity; $A_0 l$ for temperature, and $B_0 l\varkappa / D$ for concentration

$$a_1=\frac{N\lambda^2 D}{\varkappa},\quad a_2=\frac{N\lambda B_0}{A_0},\quad P_T=\frac{\nu}{\varkappa},\quad P_c=\frac{\nu}{D},\quad R_T=\frac{A_0\beta_1 g l^4}{\nu\varkappa}$$

$$R_c=\frac{B_0\beta_2 g l^4}{\nu D},\quad \lambda=\frac{k_T}{T},\quad N=\frac{T(\partial\mu/\partial c)_{p,T}}{c_p},\quad \beta_1=-\frac{1}{\rho_0}\left(\frac{\partial\rho}{\partial T}\right)_{p,c}$$

$$\beta_2=-\frac{1}{\rho_0}\left(\frac{\partial\rho}{\partial c}\right)_{p,T}$$

Here P_T and P_c are the Prandtl and Schmidt numbers; R_T and R_c are the temperature and diffusion Rayleigh numbers; $\varkappa$, D and $k_T D$ are the coefficients of thermal diffusivity, diffusion, and thermal diffusion, respectively; N is the thermodynamic coefficient; c_p is the specific heat at constant pressure; μ is the chemical potential; β_1 and β_2 are the coefficients of thermal and concentration expansion, respectively.

Boundary conditions at the solid surfaces are of the form

$$f=Df=\tau=\xi=0 \qquad \text{for} \quad z=\pm{}^1/_2 \tag{1.3}$$

Assuming that the dependence of the solutions of Eqs. (1.2) is of the form $e^{\sigma t}$, we multiply each of the equations by $f(z)$, $\tau(z)$ and $\xi(z)$, respectively, and integrate from $-{}^1/_2$ to ${}^1/_2$. As the result we obtain for the amplitude of perturbations a system of homogeneous linear algebraic equations. Equating the determinant of this system to zero, we obtain for the decrement σ the characteristic equation

$$\sigma^3+S_1\sigma^2+S_2\sigma+S_3=0 \tag{1.4}$$

where coefficients S_1, S_2 and S_3 are expressed in terms of the problem parameters. From the theory of algebraic equations [5] we know that the condition for the existence of only negative real roots of the polynomial (1.4) is

$$S_3=0 \tag{1.5}$$

while that for the existence of a pair of purely imaginary roots (i.e. the condition of indifferent oscillatory equilibrium) [5] is

$$S_3>0,\qquad S_1S_2-S_3=0 \tag{1.6}$$

The characteristic frequency of indifferent oscillatory equilibrium is then

$$\omega^2=S_2 \tag{1.7}$$

In the case of stationary and oscillatory instability the relationships for the criteria R_T and R_c are derived from conditions (1.5) and (1.6). They are

$$(1+\alpha)R_T^{(s)}+(1+a_1+a_1/\alpha)R_c^{(s)}=\gamma_0 \qquad (\alpha=-\lambda\beta_2/\beta_1)$$

$$P_c[\gamma P_TP_c+(1+a_1)P_c-\alpha P_T]R_T^{(0)}+P_T[\gamma P_TP_c+P_c-a_1P_c/\alpha]R_c^{(0)}=$$

$$=\gamma_1^{-1}[P_T+(1+a_1)P_c][(1+\gamma P_c)(1+\gamma P_T)+\gamma a_1P_c] \tag{1.8}$$

$$\gamma_0=\frac{[f,(D^2-k^2)^2f]}{[f,\tau_k]},\qquad \gamma=-\frac{[f,(D^2-k^2)^2f][\tau_k,\tau_k]}{k^2[f,\tau_k][f,(D^2-k^2)f]}$$

$$\gamma_1=-\frac{[\tau_k,\tau_k]}{k^2[f,(D^2-k^2)f]},\qquad \tau=\frac{(1-a_2)\tau_k}{k^2},\qquad [uv]\equiv\int_{-1/2}^{1/2}uv\,dz$$

where superscripts (s) and (0) relate to stationary and oscillatory instabilities, respectively.

The straight lines in (1.8) intersect in the R_TR_c-plane at the point determined by coordinates

$$R_T^* = -\gamma_1^{-1} \frac{[1+\gamma P_T + \gamma(1+a_1) P_c] a_1/\alpha + [(1+a_1)(1+\gamma P_c + \gamma a_1 P_c) + \gamma a_1 P_T]}{P_T(1+\alpha) - P_c(1+a_1+a_1/\alpha)}$$

$$R_c^* = \gamma_1^{-1} \frac{(1+\gamma P_T + \gamma a_1 P_c) + \alpha[1+\gamma P_T + (1+a_1)\gamma P_c]}{P_T(1+\alpha) - P_c(1+a_1+a_1/\alpha)} \tag{1.9}$$

The frequency of indifferent stability oscillations expressed in terms of $R_T^{(0)}$ and R_T^* is

$$\omega^2 = \left(\frac{[f, \tau_k] k^2}{[\tau_k, \tau_k]}\right)^2 \gamma_1 \frac{(1+a_1+a_1/\alpha) P_c - (1+\alpha) P_T}{P_T(\gamma P_c P_T + P_T - P_c a_1/\alpha)} (R_T^{(0)} - R_T^*) \tag{1.10}$$

(and similarly in terms of $R_c^{(0)}$ and R_c^*).

So far the exact solution was used in the analysis and derivation of all results on the assumption that functions f, τ and ξ are exact solutions of the system of equations (1.2).

For given physical constants and parameters P_T, P_c, a_1, a_2 and R_c the stationary and the oscillatory Rayleigh numbers R_T are determined by the stability conditions (1.8) as functions of the wave number k.

To determine the minimum value of $R_T(k_0)$ which, in fact, is the criterion of convection instability and k_0 is the periodicity in the horizontal plane, it is necessary to calculate the values of integrals (*). For the approximate calculation we use the Bubnov-Galerkin method with the same approximating functions for both the stationary and the oscillatory instability.

The expressions of criteria (1.8) generally contain many parameters, which complicates the analysis and computations, hence only a few particular cases will be considered.

2. In the equations of convection in a binary mixture we shall neglect the overlapping effects of thermal diffusion and of diffusive thermal conductivity ($\lambda = 0$). The conditions of monotonic and oscillatory instability are then written as

$$R_T^{(s)} + R_c^{(s)} = \gamma_0 \tag{2.1}$$

$$P_c^2(1+\gamma_T P_T) R_T^{(0)} + P_T^2(1+\gamma P_c) R_c^{(0)} = \gamma_1^{-1}(P_T + P_c)(1+\gamma P_c)(1+\gamma P_T)$$

The coordinates of the bifurcation point in the $R_T R_c$-plane are

$$R_T^* = -\gamma_1^{-1} \frac{1+\gamma P_c}{P_T - P_c}, \qquad R_c^* = \gamma_1^{-1} \frac{1+\gamma P_c}{P_T - P_c} \tag{2.2}$$

and the frequency of indifferent stability are defined by

$$\omega^2 = \left(\frac{[f, \tau_k] k^2}{[\tau_k, \tau_k]}\right)^2 \frac{1}{P_T^2} \left\{\frac{R_T^{(0)}}{R_T^*} - 1\right\} \tag{2.3}$$

The results of calculations for various ratios of $P_c / P_T = \varkappa / D$ are shown in Fig. 1, where straight lines 1—5 are branches of oscillatory instability for P_c=2, 5, 10, 0.05, 0.01, respectively, (with $P_T = 1$). It will be seen from the stability diagrams that the lines are similar to those obtained in [1]. The method and the analysis of calculation are given in the Appendix.

3. It has been assumed so far that the constant gradients A_0 of temperature and B_0 of concentration which define, respectively, the thermal and the diffusion streams q and

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*) Obviously R_T may be considered as given and the critical value of the diffusion Rayleigh number calculated R_c.

j, are independently specified. However in experiments the conditions are in the main such that $j = 0$. In that case the gradients A_0 and B_0 are interrelated (the heat flux q is then nonzero and is determined by the conditions of heating)

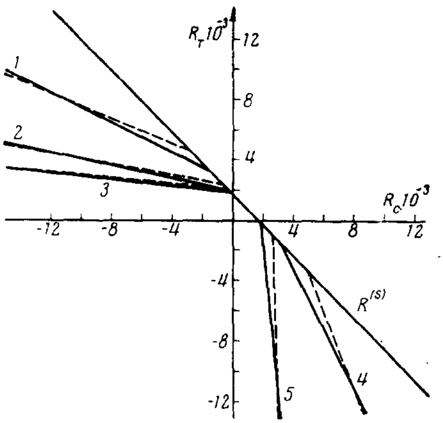

Fig. 1

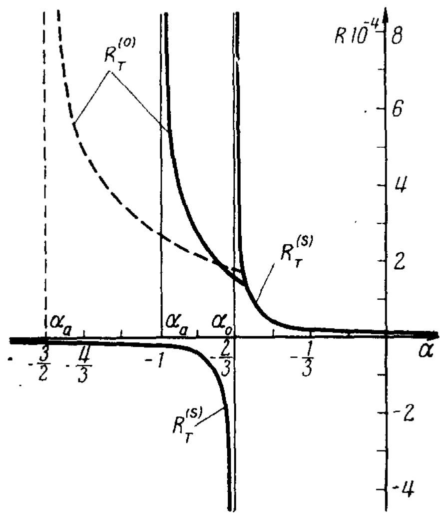

Fig. 2

$$P_T R_c = \alpha R_T P_c \qquad (\text{or} \quad \lambda A_0 = -B_0) \tag{3.1}$$

The critical Rayleigh numbers for monotonic and oscillatory instability are then defined by

$$R_T^{(s)} = \gamma_0 \frac{P_T}{(P_T + a_1 P_c) + \alpha [P_T + (1 + a_1) P_c]} \tag{3.2}$$

$$R_T^{(0)} = \gamma_1^{-1} \frac{P_T + (1 + a_1) P_c}{P_c^2 [1 + \gamma (1 + \alpha) P_T]} [(1 + \gamma P_c)(1 + \gamma P_T) + \gamma a_1 P_c]$$

Figure 2 shows $R_T^{(s)}$ and $R_T^{(0)}$ as functions of the thermal diffusion parameter α for constant P_T, P_c, a_1 ($P_T = P_c = a_1 = 1$). The curves of $R_T^{(s)}$ and $R_T^{(0)}$ intersect at $\alpha = \alpha^*$ with α^* defined by

$$\alpha^* = -\frac{(P_T + a_1 P_c)(1 + \gamma P_T + \gamma a_1 P_c) + \gamma a_1 P_c^2}{[P_T + (1 + a_1) P_c](1 + \gamma P_T + \gamma a_1 P_c) + \gamma (1 + a_1) P_c^2} \tag{3.3}$$

The frequency of indifferent stability oscillations

$$\omega^2 = \left(\frac{[f, \tau_k] k^2}{[\tau_k, \tau_k]} \right)^2 \frac{[P_T + (1 + a_1) P_c](1 + \gamma P_T + \gamma a_1 P_c) + \gamma (1 + a_1) P_c^2}{P_c^2 P_T [1 + \gamma (1 + \alpha) P_T]} (\alpha^* - \alpha) \tag{3.4}$$

vanishes at the point of bifurcation (3.3).

It follows from (3.2) that the vertical lines

$$\alpha = \alpha_0, \quad \alpha = \alpha_a \qquad \left(\alpha_0 = -\frac{P_T + a_1 P_c}{P_T + (1 + a_1) P_c}, \ \alpha_a = -\left(1 + \frac{1}{\gamma P_T} \right) \right) \tag{3.5}$$

are asymptotes: the first is that of the two branches of the hyperbola defining the monotonic instability (the axis of abscissas is the second asymptote), and the second is that of the oscillatory instability branch, i.e. this branch is bounded by the bifurcation point α^* on one side and on the other by the asymptote $\alpha = \alpha_a$.

The instability relative to monotonic perturbations is defined by two branches and can occur with heating from below, as well as from above:

for $\alpha > \alpha_0$ the upper branch $R_T^{(s)} > 0$ corresponds to heating from below;

for $\alpha < \alpha_0$ the lower branch $R_T^{(s)} < 0$ corresponds to heating from above.

The analysis of calculation is given in the Appendix.

4. One of the assumptions made in the derivation of Eqs. (1.2) was that of smallness of the pressure gradient in a binary mixture. There are, however, cases in which, as will be shown in the following, this assumption is unacceptable, since the pressure gradient effects become of the same order of magnitude as those of temperature and concentration.

In such cases the equations of convection in a binary mixture with the pressure gradient taken into account only in the equilibrium state, written in dimensionless variables, are of the form

$$\frac{\partial \mathbf{V}}{\partial t} = -\nabla p' + \Delta \mathbf{V} + (R_T T' + R_c C')\boldsymbol{\gamma} + \left(\frac{\zeta}{\nu} + \frac{1}{3}\right)\nabla \operatorname{div} \mathbf{V}$$

$$P_T \frac{\partial T'}{\partial t} = (1 - \chi_0)\mathbf{V}\boldsymbol{\gamma} + (1 + a_1)\Delta T' + a_2 \Delta C' \qquad (4.1)$$

$$P_c \frac{\partial C'}{\partial t} = \mathbf{V}\boldsymbol{\gamma} + \frac{a_1}{a_2}\Delta T' + \Delta C'$$

$$\operatorname{div} \mathbf{V} = -l(L_1 + L_2 + L_3)\mathbf{V}\boldsymbol{\gamma} + lL_1 P_T \frac{\partial T'}{\partial t} + lL_3 P_c \frac{\partial C'}{\partial t}$$

$$\left(\chi_0 = \frac{\beta_1 T g}{A_0 c_p} = \frac{L_2}{L_1}\left(1 - \frac{c_v}{c_p}\right), \quad L_1 = \beta_1 A_0, \quad L_2 = g\left(\frac{\partial \rho}{\partial p}\right)_{T,c}, \; L_3 = \beta_2 B_0\right)$$

Here, as in the equations of convection for a pure compressible fluid, we have the Schwarzschild criterion χ_0 and, also, three new parameters lL_1, lL_2 and lL_3 which define, respectively, thermal expansion, compressibility and expansion of the binary mixture resulting from the change of concentration.

After transformation, using, as previously, the variational method, we obtain a system of equations the analysis of which with respect to stability by the method of Routh-Hurvitz yields in the general case the condition of monotonic and oscillatory instability (*)

$$(1 + \alpha - \chi_0) R_T^{(s)} + (1 + a_1 + a_1/\alpha) R_c^{(s)} = \gamma_0$$

$$P_c[\gamma P_c P_T + (1 + a_1)P_c - \alpha P_T - \chi_0 P_c(1 + a_1 + \gamma P_T)] R_T^{(0)} +$$

$$+ P_T[\gamma P_c P_T + P_T - a_1 P_c/\alpha] R_c^{(0)} = \qquad (4.2)$$

$$= \gamma_1^{-1}[P_T + (1 + a_1)P_c][(1 + \gamma P_c)(1 + \gamma P_T) + a_1 P_c]$$

Let us now consider the case – most interesting from the experimental point of view – in which the gradients A_0 and B_0 of temperature and concentration, respectively, are bound at equilibrium by the condition $\mathbf{j} = 0$. Taking into account the pressure gradient,

*) The terms – odd with respect to z – appearing in the equation of motion are small owing to the smallness of parameters L_1, L_2, L_3. In the following we take into consideration only those parameters which have singularities in the neighborhood of the diffusion line, and, since β_1, β_2 and $(\partial\rho/\partial p)_{T,c}$ do not have such singularities, the result will correspond to the compressibility being taken into account only in the thermal conductivity equation.

we have

$$B_0 + \lambda A_0 = -g\beta_2 / \mu_c \quad (\mu_c \equiv (\partial\mu / \partial c)_{p,T}) \tag{4.3}$$

or in dimensionless form

$$P_T R_c = \alpha P_c R_T - \varepsilon P_T, \quad \varepsilon = \frac{(g\beta_2 l^2)^2}{\nu D \mu_c}$$

For monotonic and oscillatory perturbations the expressions for the critical Rayleigh numbers are in this case of the form

$$R_T^{(s)} = P_T \frac{\gamma_0 + \varepsilon(1 + a_1 + a_1/\alpha)}{[(1-\chi_0) P_T + a_1 P_c] + \alpha [P_T + (1 + a_1) P_c]}$$

$$R_T^{(0)} = \frac{1}{\gamma_1 P_c^2 F} \{[P_T + (1 + a_1) P_c] [(1 + \gamma P_c)(1 + \gamma P_T) + \gamma a_1 P_c] +$$
$$+ \varepsilon P_T (\gamma P_c P_T + P_T - a_1 P_c / \alpha)\}$$
$$F = [1 + (1 + \alpha) \gamma P_T] - \chi_0 (1 + a_1 + \gamma P_T) \tag{4.4}$$

The frequency of indifferent stability oscillations are

$$\omega^2 = \left(\frac{[f, \tau_k] k^2}{[\tau_k, \tau_k]}\right)^2 \frac{(\alpha_1^* - \alpha)(\alpha_2^* - \alpha)}{P_T P_c^2 \alpha F} \tag{4.5}$$

Here α_1^* and α_2^* are bifurcation points

$$\alpha_{1,2}^* = \frac{-m \pm \sqrt{m^2 - 4n\varepsilon\gamma_1 P_T P_c a_1}}{2n}$$

$$m = (1 - \chi_0) \{[P_T + (1 + a_1) P_c] [\gamma_1 \varepsilon P_T - 1 - \gamma (P_T + a_1 P_c)] + P_c (1 + \gamma P_T)\} +$$
$$+ \chi_0 \{\gamma_1 \varepsilon P_T^2 (1 - \gamma P_c) - a_1 P_c [1 + \gamma P_T + \gamma (1 + a_1) P_c]\} \tag{4.6}$$
$$n = \gamma_1 \varepsilon P_T^2 - \gamma [P_T + (1 + a_1) P_c]^2 - [P_T + (1 + a_1) P_c - \gamma P_c P_T]$$

(m and n are of the same sign and $\alpha_1^* < 0$, $\alpha_2^* < 0$, $\alpha_2^* < \alpha_1^*$).

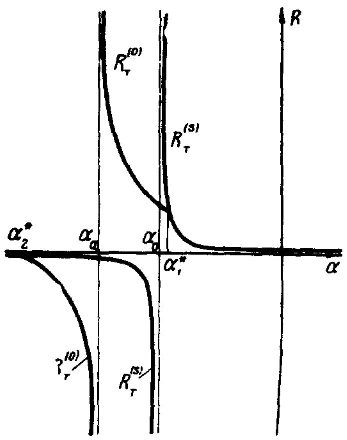

Fig. 3

It is seen from (4.4) that the vertical lines

$$\alpha = \alpha_0 \text{ and } \alpha = \alpha_a$$

$$\left(\alpha_0 = -(1 - \chi_0)\left(1 + \frac{1}{\gamma P_T}\right) + \frac{\chi_0 a_1}{\gamma P_T}\right.$$

$$\left.\alpha_a = -\frac{(1 - \chi_0) P_T + a_1 P_c}{P_T + (1 + a_1) P_c}\right) \tag{4.7}$$

are asymptotes: the first is that of the two branches defining monotonic instability, the second is that of the two indifferent oscillatory instability branches emanating from the two different bifurcation points α_1^* and α_2^* (Fig. 3). Thus in this case both the oscillatory and the monotonic instabilities are possible with heating from below and from above.

5. The expressions for the conditions of convection onset in a binary mixture contain derivatives of physical magnitudes which in the vicinity of the diffusion line have singularities. Hence the convection criteria can substantially vary in the critical region. The critical line is defined in terms of p, T, C [6] as

follows:

$$\left(\frac{\partial \mu}{\partial C}\right)_{p,\,T} = 0, \quad \left(\frac{\partial^2 \mu}{\partial C^2}\right)_{p,\,T} = 0 \tag{5.1}$$

In the critical region

$$\left(\frac{\partial \mu}{\partial C}\right)_{p,\,T} \sim t \qquad \left(t = \frac{T - T_*}{T_*}\right)$$

Here T_* (p) is the critical temperature of diffusion along the line of critical points at fixed pressure p. In accordance with the definitions

$$\alpha \sim \lambda \sim \left(\frac{\partial \mu}{\partial C}\right)^{-1}_{p,\,T}, \quad D \sim \left(\frac{\partial \mu}{\partial C}\right)_{p,\,T}, \quad P_c \sim \left(\frac{\partial \mu}{\partial C}\right)^{-1}_{p,\,T}, \quad \varepsilon \sim \left(\frac{\partial \mu}{\partial C}\right)^{-2}_{p,\,T}$$

Let us rewrite the derived criteria of convection onset, taking this into consideration and denoting by subscript (a) the thermodynamic parameters away from T_*.

In the general case from (1.8)–(1.10) we obtain for $t \to 0$ the following asymptotic expressions:

$$\alpha_{(a)} R_T^{(s)} + (1 + a_1)\, R_{c(a)}^{(s)} = \gamma_0 t \to 0$$

$$\left[1 + a_1 + \gamma P_T - P_T \left(\frac{\alpha}{P_c}\right)_{(a)}\right] R_T^{(0)} + \gamma P_T^2 \left(\frac{R_c^{(0)}}{P_c}\right)_{(a)} = \gamma_0 (1 + a_1)(1 + a_1 + \gamma P_T)$$

$$\omega^2 = \left(\frac{[f, \tau_k]\, k^2}{[\tau_k, \tau_k]}\right)^2 \gamma_1 \frac{(1 + a_1) P_{c(a)} - \alpha_{(a)} P_T}{\gamma P_T{}^2 P_{c(a)}} \left(R_T^{(0)} - R_T^*\right) \tag{5.2}$$

$$R_T{}^* = \gamma_0 (1 + a_1)\left[1 - \frac{P_T}{1 + a_1}\left(\frac{\alpha}{P_c}\right)_{(a)}\right]^{-1}, \quad R_{c(a)}^* = -\gamma_0 \alpha_{(a)}\left[1 - \frac{P_T}{1 + a_1}\left(\frac{\alpha}{P_c}\right)_{(a)}\right]^{-}$$

Neglecting the overlapping effects, we obtain from (2.1)–(2.3)

$$R_T^{(s)} + R_{c(a)}^{(s)} t^{-1} = \gamma_0, \quad R_T^{(0)} + \frac{\gamma P_T}{1 + \gamma P_T} P_T \left(\frac{R_c^{(0)}}{P_c}\right)_{(a)} = \gamma_0$$

$$\omega^2 = \left(\frac{[f, \tau_k]\, k^2}{[\tau_k, \tau_k]}\right)^2 \frac{1}{P_T{}^2}\left\{\frac{R_T^{(0)}}{R_T{}^*} - 1\right\}, \quad R_T{}^* = \gamma_0, \quad R_{c(a)}^* \sim -t^2 \to 0 \tag{5.3}$$

The monotonic instability condition (5.3) shows that the admissible gradient of concentration $R_c{}^{(s)} \sim t \to 0$, consequently the branch of monotonic instability degenerates

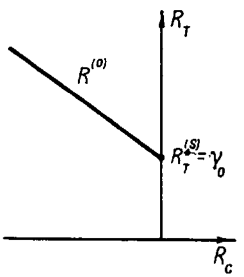

Fig. 4

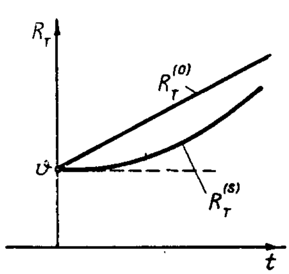

Fig. 5

in the $R_T R_c$-plane into point $R_T{}^{(s)} = \gamma_0$, i.e. we have the condition of stationary instability in a pure fluid. The oscillatory instability branch begins at point $R_T{}^* = \gamma_0$, $R_c{}^* = O$ $(T = T_*)$ and occurs for $R_c{}^{(0)} \leqslant 0$ and $R_T{}^{(0)} \geqslant \gamma_0$ (Fig. 4). The most interesting case (from the point of view of comparison with expreimental data) is that of $j = 0$. In that case for the monotonic and the oscillatory Rayleigh numbers, and for the frequency of indifferent stability oscillations in the neighborhood of the critical diffusion line in a binary mixture we obtain from (4.4)–(4.6) the expressions

$$R_T^{(s)} = P_T\,(\varepsilon/\alpha P_c)_{(a)} + \gamma_0 \frac{P_T}{(1+a_1)\,(\alpha P_c)_{(a)}}\,t^2 \qquad (5.4)$$

$$R_T^{(0)} = P_T\,(\varepsilon/\alpha P_c)_{(a)} + \gamma_1^{-1} \frac{(1+a_1)(1+a_1+\gamma P_T)}{P_T\,(\alpha P_c)_{(a)}}\,t, \quad \omega \sim |t| \to 0$$

Thus both the oscillatory and the monotonic Rayleigh numbers, although varying to different laws, tend to the same constant $P_T\,(\varepsilon/\alpha P_c)_{(a)}$ when approaching T_* (Fig. 5). Since, however, simultaneously $\omega \to 0$, there remains on the line of critical points only the stationary instability with the critical Rayleigh number

$$R_T^{(s)} = P_T\,(\varepsilon/\alpha P_c)_{(a)}, \quad \text{or} \quad \vartheta \equiv -\frac{A_0/\lambda}{g\beta_2} = 1 \qquad (5.5)$$

i.e. a new criterion appears at the critical point of a binary solution.

If, on the other hand, the pressure gradient in the equilibrium state is taken into account, then (3.2)–(3.4) yield an entirely different result

$$R_T^{(s)} \sim t^2, \quad R_T^{(0)} \sim t, \quad \omega^2 = \left(\frac{[f,\,\tau_k]\,k^2}{[\tau_k,\,\tau_k]}\right)^2 \left(\frac{1+a_1}{P_T}\right)^2 \left(\frac{\alpha^*}{\alpha} - 1\right) \qquad (5.6)$$

i.e. both kinds of instability are present but the values of criteria tend to vanish with the approach to the critical point. This shows that the criterion of convection onset is considerably altered when the effects related to pressure gradient in the critical region of a binary solution in equilibrium are taken into account.

This is similar to the case of a pure fluid [7], when compressibility is taken into consideration only in the equations of thermal conductivity (only in equilibrium). In this case the critical Rayleigh number in the neighborhood of the liquid-vapor critical point does not tend to zero but to a constant defined by the Schwarzschild criterion χ_0.

In a pure fluid this is related to compressibility characteristics in proximity to T_*, while the singularity in the critical region of a binary mixture has a corresponding susceptibility μ_c^{-1}. This criterion also defines the convection instability in binary mixture layers of considerable heights similarly to the Schwarzschild criterion in the case of pure fluid.

It should be pointed out that here we have considered only the critical point of mixing, where the compressibility of a binary mixture is low.

However, as shown in [8], the compressibility of a binary solution also has a singularity along the line of critical points of evaporation of infinitely diluted solutions and in the neighborhood of that line intersection with the azeotrope. This makes it necessary to take compressibility systematically into account in the equations of hydrodynamics and, because of the hydrostatic effect, the dependence of the coefficients in the equations on coordinates, as was done in the case of a pure compressible fluid in [9].

Appendix. The numerical calculation of integrals in expressions for the problem eigenvalues is carried out by the Bubnov-Galerkin method with the substitution of certain approximating functions satisfying the problem boundary conditions for the exact solutions of equations.

In the case considered here even the second approximation (two-term expansion) to the solution of the oscillatory instability problem presents considerable difficulties. The characteristic equation (1.4) instead of being cubic becomes then one of the sixth power, which leads to a very cumbersome expression for the related condition of oscillatory instability.

Here, as in the case of a pure fluid, the condition of monotonic instability is not complicated, since the sixth-order determinant S_6 reduces to a third-order lattice determinant from which follows condition (1.8) but with a different right-hand side which coincides with the corresponding expression for a pure fluid.

Thus, in the case of monotonic instability of a binary mixture, the use of approximating functions $f_1 = (1 - 4z^2)^2$ and $f_2 = 1 + \cos 2\pi z$, checked in [7], for (calculating) the vertical velocity amplitude together with related solutions of equations of thermal conductivity and diffusion yields for γ_0 values close to the exact ones of 1707.8 and 1802, respectively, (the exact value $\gamma_0 = 1707.8$ for $k_0 = 3.1$ is the limit case of pure fluid when $B_0 = 0$).

The integrals appearing in (1.8) in the expressions for γ_0, γ and γ_1 are:

1) for $f_1 = (1 - 4z^2)^2$

$$[f, f] = \frac{128}{315}, \quad [f, (D^2 - k^2)^2 f] = \frac{128}{315}(k^4 + 24k^2 + 504), \quad [f, (D^2 - k^2) f] = -\frac{128}{315}(12 + k^2)$$

$$[f, \tau_k] = \frac{128k^{-9}}{315}\{k^5(k^4 - 12k^2 + 504) + 5040(12 + k^2)[6k - (12 + k^2)\, th\, {}^1\!/\!_2 k]\}$$

$$[\tau_k, \tau_k] = \frac{128k^{-9}}{315}\{k(k^8 - 24k^6 + 4914k^4 + 362\,880) + (12 + k^2)(k + 4shk)(1 + chk)^{-1} + $$
$$+ 20\,160(12 + k^2)[6k - (18 + k^2)\, th\, {}^1\!/\!_2 k]\}$$

2) for $f_2 = 1 + \cos 2\pi z$

$$[f, f] = {}^3\!/\!_2, \quad [f, (D^2 - k^2)^2 f] = {}^3\!/\!_2 k^4 + 4\pi^2 k^2 + 8\pi^4, \quad [f, (D^2 - k^2) f] = -{}^1\!/\!_2(3k^2 + 4\pi^2)$$

$$[f, \tau_k] = 1 - \left(\frac{4\pi^2}{4\pi^2 + k^2}\right)^2 \frac{2}{k}\, th\, {}^1\!/\!_2 k + \frac{1}{2}\frac{k^2}{4\pi^2 + k^2}$$

$$[\bar{\tau}_k, \tau_k] = 1 - \frac{64\pi^4(2\pi^2 + k^2)}{(4\pi^2 + k^2)^3}\frac{2}{k}\, th\, {}^1\!/\!_2 k + \frac{1}{2}\left(\frac{k^2}{4\pi^2 + k^2}\right)^2 + \left(\frac{4\pi^2}{4\pi^2 + k^2}\right)^2 \frac{k + shk}{k(1 + chk)}$$

If the approximate eigenvalues of the problem obtained with the use of these same two approximating functions are close to each other, one can reasonably expect that they are close to the exact eigenvalue of the problem.

First of all, from the condition of coincidence of the point of intersection of straight lines defined by (1.8) with the point at which $\omega = 0$ is satisfied, we obtain at the bifurcation point $\gamma\gamma_1^{-1} = 1707.8$ for $k_0 = 3.1$ (using f_1) and 1802 for $k_0 = 3.1$ (using f_2) (this follows, also, from the fact that $\gamma\gamma_1^{-1} = \gamma_0$ and that at the bifurcation point it is equal to the value taken along the monotonic branch of stability).

The oscillatory Rayleigh number $R_T^{(0)}(k)$ calculated by Eq. (2.1) or (3.2) as a function of k with f_2 as the approximating function is represented by a smooth curve with a single minimum at $k_0 = 3.1$ for all values of parameters P_c, P_T, R_c, a_1.

Values of the oscillatory instability criterion $R_T^{(0)}$ and of periodicity in the horizontal plane k_0 calculated by the second of Eqs. (2.1) with f_1 as the approximating function are close to those calculated with f_2 as the trial function (Fig. 1) (*).

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*) In all diagrams the results of calculations with f_1 and f_2 taken as the approximating functions are shown by solid and hatched lines respectively.

However, in spite of the closeness criterions obtained with functions f_1 and f_2, the values of γ and γ_1 in (1.8) and the critical frequencies ω vary considerably. Thus for $P_T = 1$, $P_c = 2$ and $k_0 = 3.1$ we have:

for $f_1 = (1 - 4z^2)^2$

$$\gamma = -122.56, \quad \gamma_1 = -0.072, \quad \omega^2 = 0.062 \quad \text{for } \overset{(0)}{R_c} = 6000$$
$$\omega^2 = 0.119 \quad \text{for } \overset{(0)}{R_c} = 10{,}000$$

for $f_2 = 1 + \cos 2\pi z$

$$\gamma = 1.937, \quad \gamma_1 = 0.00107, \quad \omega^2 = 108.35 \quad \text{for } \overset{(0)}{R_c} = 6000$$
$$\omega^2 = 253.98 \quad \text{for } \overset{(0)}{R_c} = 10{,}000$$

It should be noted here that the curve of $R_T^{(0)}(k)$ calculated with f_1 (as distinct from f_2) as the trial function has an additional minimum which, for any values of parameters P_T and P_c, and $k = 3.5$ (the same for various values of parameters), appears at some distance from the bifurcation point and deepens with increasing distance from that point. For high values of parameters this minimum is deeper than the primary one at $k = k_0$. In the neighborhood of this minimum function $R_T^{(0)}(k)$ becomes discontinuous (for $P_T = 1$, $P_c = 0.1$, $R_c^{(0)} = 10\ 000$ and for $k = 3.4$ function $R_T^{(0)}(k)$ undergoes a change of the order of twenty). Furthermore, for $P_c < P_T = 1$ at high $R_c^{(0)}$ the minimum k_0 tends to decrease with increasing distance from the bifurcation point along the oscillation branch (thus when $P_c = 0.1$ and $R_c^{(0)} = 10^4$, then $k_0 = 2.7$). Since P_c and P_T are symmetric in the criterion, all this holds, also, in the opposite case.

Similar results are obtained for the criterion of oscillatory instability by (3.2) with f_1 as the trial function. Furthermore here, owing to the considerable difference between the value of γ calculated with f_1 as the trial function and that calculated with f_2, the position of the asymptote of the oscillatory instability branch is substantially affected (see Fig. 2).

On the basis of calculations presented here it is possible to come to the conclusion that the Bubnov-Galerkin method, when used for computing the criterion of oscillatory instability, necessitates an even more careful selection of the approximating function than in the case of monotonic instability. As shown on the example of an unfortunate choice of the approximating function ($f_1 = (1-4z^2)^2$), although yielding a good approximate value of the criterion, leads in many cases to considerable errors in the calculation of γ, γ_1 and ω^2.

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